Why do we study metric geometries? It is because many of the concepts in the synthetic approach which must be added are already present in the metric geometry approach. This happens because we can transfer questions about a line ℓ in $\mathcal{L}$, to the real numbers $\mathbb{R}^2$ by using a ruler f. In $\mathbb{R}$ we understand concepts like "between" and so can transfer them back (via f^{-1}) to ℓ. This is the advantage of the metric approach alluded to in the beginning of the section. After we have more background, we will return to the question of a synthetic versus metric approach to geometry.

28. Denote by $\{\mathcal{S},\mathcal{L},d\}$ a metric geometry, let $P\in\mathcal{S}$ denote arbitrary point, $p\in\mathcal{L}$ such that $P\in p$, and let $r\in\mathbb{R}$. Show that on line p there exist at least one point Q such that $d(P,Q)=r$.

29. Let $\{\mathcal{S},\mathcal{L}\}$ be an incidence geometry. Assume that for each line $\ell\in\mathcal{L}$ there exists a bijection $f_\ell:\ell\to\mathbb{R}$. Show that then there is a distance d such that $\{\mathcal{S},\mathcal{L},d\}$ is a metric geometry and each $f_\ell:\ell\to\mathbb{R}$ is a ruler.

30. Let d^* denote distance function on $\mathbb{R}^2$ which is defined on the following way

$$d^*(P,Q)=\begin{cases} d_E(P,Q), & \text{if } d_E(P,Q)\le 1\\ 1, & \text{if } d_E(P,Q)>1\end{cases}$$

Prove that there is no incidence geometry on $\mathbb{R}^2$ such that $\{\mathbb{R}^2,\mathcal{L},d^*\}$ is a metric geometry. (Thus not every distance gives a metric geometry.)

31. If $\{\mathcal{S},\mathcal{L},d\}$ is a metric geometry and $P\in\mathcal{S}$, prove that for any $r>0$ there is a point in $\mathcal{S}$ at distance r from P.

32. Define the max distance (or supremum distance), d_S, on $\mathbb{R}^2$ by

$$d_S(P,Q)=\max\{|x_1-x_2|,|y_1-y_2|\}$$

where $P(x_1,y_1)$ and $Q(x_2,y_2)$.

(i) Show that d_S is a distance function.

(ii) Show that $\{\mathbb{R}^2,\mathcal{L},d_S\}$ is a metric geometry.

33. In a metric geometry $\{\mathcal{S},\mathcal{L},d\}$ if $P\in\mathcal{S}$ and $r>0$, then the circle with center P and radius r is $\mathcal{C}=\{Q\in\mathcal{S}\mid d(P,Q)=r\}$. Draw a picture of the circle of radius 1 and center $(0,0)$ in the $\mathbb{R}^2$ for each of the distances d_E, d_T, and d_S.

34. Let $\{\mathcal{S},\mathcal{L},d\}$ be a metric geometry, let $P\in\mathcal{S}$, let $\ell\in\mathcal{L}$ with $P\in\ell$, and let $\mathcal{C}=\{Q\in\mathcal{S}\mid d(P,Q)=r\}$ be a circle with center P. Prove that $\ell\cap\mathcal{C}$ contains exactly two points.

35. Find the circle of radius 1 with center $(0,e)$ in the Poincaré Plane. Hint: As a set this circle "looks" like an ordinary circle. Carefully show this.

36. We may define a distance function for the Riemann Sphere as follows. On a great circle $\mathcal{C}$ we measure the distance $d_R(A,B)$ between two points A and B as the shorter of the lengths of the two arcs of $\mathcal{C}$ joining A to B. (Note $d_R(A,-A)=\pi$.) Prove that d_R is a distance function. Is $\{S^2,\mathcal{L}_R,d_R\}$ a metric geometry?

3 Special Coordinate Systems

Theorem
Let f be a coordinate system for the line ℓ in a metric geometry. If $a \in \mathbb{R}$ and ε is ± 1 and if we define $h_{a,\varepsilon} : \ell \to \mathbb{R}$ by

$$h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$$

then $h_{a,\varepsilon}$ is a coordinate system for ℓ.

1. Prove the previous theorem.

2. Let f be a coordinate system for the line ℓ in a metric geometry. Define $h_{a,\varepsilon} : \ell \to \mathbb{R}$ by $h_{a,\varepsilon}(P) = \varepsilon(f(P) - a)$ (where $a \in \mathbb{R}$, and ε is ± 1). Explain and geometrically show difference between
(i) f and $h_{0,-1}$;
(ii) f and $h_{a,1}$.

3. Let ℓ be a line in a metric geometry and let A and B be points on the line. Show that there is a coordinate system g on ℓ with $g(A) = 0$ and $g(B) > 0$.

Definition (coordinate system with A as origin and B positive.)
Let $\ell = \ell(A, B)$. If $g : \ell \to \mathbb{R}$ is a coordinate system for ℓ with $g(A) = 0$ and $g(B) > 0$, then g is called a coordinate system with A as origin and B positive.

4. In the Euclidean Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,-5)$; ii. $P(2,3)$, $Q(4,0)$.

5. In the Poincaré Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,1)$; ii. $P(2,3)$, $Q(-1,6)$.

6. In the Taxicab Plane find a ruler f with $f(P) = 0$ and $f(Q) > 0$ for the given pair P and Q:

i. $P(2,3)$, $Q(2,-5)$; ii. $P(2,3)$, $Q(4,0)$.

It is reasonable to ask if there are any other operations (besides reflection and translation) that can be done to a coordinate system to get another coordinate system; that is, is every coordinate system of the form $h_{a,\varepsilon}$?

7. If ℓ is a line in a metric geometry and if $f : \ell \to \mathbb{R}$ and $g : \ell \to \mathbb{R}$ are both coordinate systems for ℓ, show that then there is an $a \in \mathbb{R}$ and an $\varepsilon = \pm 1$ with $g(P) = \varepsilon(f(P) - a)$ for all $P \in \ell$.

8. Prove that a line in a metric geometry has infinitely many points.

9. Let P and Q be points in a metric geometry. Show that there is a point M such that $M \in p(P,Q)$ and $d(P,M) = d(M,Q)$.

10. Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry and $Q \in \mathcal{S}$. If ℓ is a line through Q show that for each real number $r > 0$ there is a point $P \in \ell$ with $d(P,Q) = r$. (This says that the line really extends indefinitely.)

11. Let $g : \mathbb{R} \to \mathbb{R}$ by $g(s) = s/(|s|+1)$. Show that g is injective.

12. Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry. For each $\ell \in \mathcal{L}$ choose a ruler f_ℓ. Define the function $\overline{d}$ by $\overline{d}(P,Q) = |g(f_\ell(P)) - g(f_\ell(Q))|$ where $\ell = \ell(P,Q)$ and g is as in Problem 11. Show that $\overline{d}$ is a distance function.

13. In Problem 12 show that $\{\mathcal{S}, \mathcal{L}, \overline{d}\}$ is not a metric geometry.

A metric geometry always has an infinite number of points (Problem 8). In particular, a finite geometry cannot be a metric geometry.